Descriptive Set Theory
Lecture 24

There is also a cool "probabilistic" proof that analytic sets have the PSP via this theorem:

Theorem. Let X, Y be Polish and $f: X \to Y$ be continuous (so $f(X)$ is an analytic set). If $f(X)$ is unctbl, then $\exists$ homeomorphic copy $C \subseteq X$ of $2^{\mathbb{N}}$ s.t. $f|_C$ is 1-1. In particular, $f(C)$ is a homeomorphic copy of $2^{\mathbb{N}}$ inside $f(X)$.

Proof. One shows that a generic $K \in \mathcal{K}(X')$ has the property that $f|_K$ is 1-1, where $X' := X \setminus f^{-1}(\mathcal{U})$, where $\mathcal{U}$ is the maximal open set in Y s.t. $\mathcal{U} \cap f(X)$ is ctbl. This X' is perfect, so a generic $K \in \mathcal{K}(X')$ is also perfect (by homework). Thus, a generic subset $K \in \mathcal{K}(X')$ is both perfect and $f|_K$ is 1-1. Hence $\exists$ one such K and $2^{\mathbb{N}} \hookrightarrow K$ by the perfect set theorem. $\square$

Baire measurability of analytic sets. Similarly, we unravel the

Banach – Mazur game: Let X be a Polish space. The unraveled Banach – Mazur game $G_u^{BM}(P)$ on $P \subseteq X \times \mathbb{N}^{\mathbb{N}}$ is:

Player 1.	U_0, y_0		U_1, y_1	. . .
Player 2.		V_0		V_1

where $y_i \in \mathbb{N}$, $U_n \subseteq_c V_n \subseteq_c U_{n+1}$ $\forall n$, $\mathrm{diam}(U_n) \to 0$.
Player 1 wins if $(x, y) \in P$ where $\{x\} = \bigcap_n U_n = \bigcap_n \overline{U_n} = \bigcap_n V_n$ and $y := (y_n)$.

Theorem. (a) If Pl. 1 has a winning str. in $G_u^{BM}(P)$ then she has a winning str. in $G^{BM}(\mathrm{proj}_X P)$, hence $\mathrm{proj}_X P \supseteq^* U \neq \emptyset$ for some open set U.
(b) If Pl. 2 has a winning str. in $G_u^{BM}(P)$ then $\mathrm{proj}_X P$ is meager.

Proof. (a) is trivial and (b) is analogous to the corresponding statement for $G^{BM}(\mathrm{proj}_X P)$, like we did with the cut-and-choose game. $\square$

Cor. Analytic (hence also coanalytic) sets are Baire measurable.
Proof. Given an analytic set A, it is enough to show that the (still analytic set) $A \setminus U(A)$ is meager. There is

a closed $F \subseteq X \times \mathbb{N}^{\mathbb{N}}$ s.t. $A \setminus U(A) = \mathrm{proj}_X F$.
As with the cut-and-choose game, the map play $\mapsto$ outcome (x,y) is contin. hence $G_u^{BM}(F)$ is a closed game, hence determined by Gale-Stewart.
If Player 1 wins, then $\mathrm{proj}_X F = A \setminus U(A)$ contains a nonempty open set contradicting the def. of $U(A)$.
Thus Player 2 wins, hence $\mathrm{proj}_X F = A \setminus U(A)$ is meager. $\square$

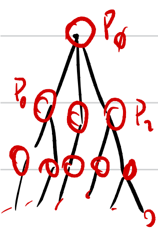
One could also show that analytic (hence also coanalytic) sets are universally measurable by using the measure isomorphism theorem and playing the unravelled Banach-Mazur game on $[0,1]$ with the Lebesgue density top (which is not Polish).

We will give an alternative gameless proof of Baire meas. and universal measurability of analytic sets using the so-called Souslin operation $\mathcal{A}$.

Souslin operation $\mathcal{A}$. $\mathcal{A}$ is an operation applied to a sequence of sets and we'll show that $\Sigma_1^1 = \mathcal{A}\,\Pi_1^0$ and that the classes of Baire meas. and univer-

sally measurable sets are closed under operation $\mathcal{A}$.
This implies that $\Sigma^1_1 \subseteq$ Baire meas. $\cap$ univ. meas.

Def. A Souslin scheme on a Polish X is just a seq $(P_s)_{s\in T}$ of subsets $P_s \subseteq X$ indexed by elements in a pruned tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$. The operation $\mathcal{A}$ applied to this Souslin scheme is



$$\mathcal{A}(P_s)_{s\in T} := \underbrace{\bigcup_{y\in[T]}}_{\text{unctbl union}} \bigcap_n P_{y|n}.$$

Prop. Let X be Polish and $(P_s)_{s\in T}$ a Souslin scheme on X.

(a) If $(P_s)_{s\in T}$ is actually a Lusin scheme, i.e. $s \subseteq t \Rightarrow P_s \supseteq P_t$ and $P_{s^\frown n} \cap P_{s^\frown m} = \emptyset$ for $n \neq m \in \mathbb{N}$, then
$$\mathcal{A}(P_s)_{s\in T} = \bigcap_{n\in\mathbb{N}} \bigcup_{s\in T,\, |s|=n} P_s.$$

(b) If $\forall s \in T$, $P_s = \bigcup_{\substack{n\in\mathbb{N}\\ s^\frown n\in T}} P_{s^\frown n}$, then $\mathcal{A}(P_s)_{s\in T} = P_\emptyset$.

Proof. (b) is immediate, and for (a), note that $\subseteq$ holds for any Souslin scheme, and for a Lusin scheme, if $x \in$ right handside, then $\forall n\ \exists\ \overset{P_{s^\frown n}\cap P_{s^\frown m}=\emptyset}{\text{unique}}\ s_n \in T, |s_n| = n$ s.t.

$x \in P_{s_n}$ and $s_n \subseteq s_{n+1}$ (by $P_s \supseteq P_{s^\frown n}$). Thus, $y := \bigcup_n s_n$ $\in [T]$ and $x \in \bigcap_n P_{y|_n}$ hence $x \in \mathcal{A}(P_s)_{s\in T}$. $\square$

Obs. WLOG, we may assume that $\forall s, n$, $P_s \supseteq P_{s^\frown n}$ because we can replace $P'_s := P_s \cap \left(\bigcup_{\substack{n\in\mathbb{N} \\ s^\frown n \in T}} P_{s^\frown n}\right)$ and still have $\mathcal{A}(P'_s)_{s\in T} = \mathcal{A}(P_s)_{s\in T}$.

But what part (c) above says is that the real complexity of the operation $\mathcal{A}$ comes from sets on the same level not being disjoint.

Prop. For a Souslin scheme $(P_s)_{s\in T}$, $\exists$ set $P \subseteq X \times \mathbb{N}^{\mathbb{N}}$ s.t. $\mathcal{A}(P_s)_{s\in T} = \mathrm{proj}_X P$. Namely: $\forall (x,y) \in X \times \mathbb{N}^{\mathbb{N}}$,

$$(x,y) \in P :\Leftrightarrow \forall n \in \mathbb{N} \;\; x \in P_{y|_n}$$
$$\Leftrightarrow \forall n \in \mathbb{N} \; \exists s \in T \text{ s.t. } s = y|_n \;\; x \in P_s$$
$$\Leftrightarrow \forall n \in \mathbb{N} \; \forall s \in T \; (s = y|_n \Rightarrow x \in P_s).$$

In particular, $\mathcal{A}\Sigma^1_1 \subseteq \underbrace{\exists^{\mathbb{N}^{\mathbb{N}}}\Sigma^1_1}_{\Sigma^1_1}$, also $\mathcal{A}\Pi^0_1 \subseteq \exists^{\mathbb{N}^{\mathbb{N}}}\Pi^0_1$ $= \Sigma^1_1$.

Proof. Clear. $\square$

Next, we show that $\mathcal{A}\Pi^0_1 = \Sigma^1_1 = \mathcal{A}\Sigma^1_1$. This will follow from the following:

Prop. Let $f : \mathbb{N}^{\mathbb{N}} \to X$ be continuous, X Polish, and put $P_s := f([s])$, so $P_s \in \Sigma^1_1$. Then $f(\mathbb{N}^{\mathbb{N}}) = \mathcal{A}(P_s)_{s \in \mathbb{N}^{<\mathbb{N}}} = \mathcal{A}(\overline{P}_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$. In particular, $\Sigma^1_1 \subseteq \mathcal{A}\Pi^0_1$, hence $\Sigma^1_1 = \mathcal{A}\Pi^0_1$.

Proof. Note that $f(\mathbb{N}^{\mathbb{N}}) = f([\emptyset]) = P_\emptyset$ and $P_s = f([s]) = f\left(\bigcup_{n \in \mathbb{N}} [s^\frown n]\right) = \bigcup_{n \in \mathbb{N}} f[s^\frown n] = \bigcup_{n \in \mathbb{N}} P_{s^\frown n}$, so $\mathcal{A}(P_s)_{s \in \mathbb{N}^{<\mathbb{N}}} = P_\emptyset = f(\mathbb{N}^{\mathbb{N}})$.

For the second equality, we only need to show $\mathcal{A}(\overline{P}_s) \subseteq \mathcal{A}(P_s)$. Fix $x \in \mathcal{A}(\overline{P}_s)$, i.e. $\exists y \in \mathbb{N}^{\mathbb{N}}$ s.t. $x \in \bigcap_{n \in \mathbb{N}} \overline{P}_{y|_n}$. We show that $x \in \bigcap_n P_{y|_n}$ using the continuity of f.

For this, it's enough to show $f(y) = x$ because then $x \in f[y|_n] = P_{y|_n}$ $\forall n \in \mathbb{N}$. Suppose $f(y) \neq x$, then $\exists$ open $U \ni f(y)$ s.t. $x \notin \overline{U}$. By continuity, $\exists n$ s.t. $f[y|_n] \subseteq U$, so $\overline{P}_{y|_n} = \overline{f[y|_n]} \subseteq \overline{U}$ and hence $x \notin \overline{P}_{y|_n}$, contradiction. $\square$

It remains to show that the class of Baire meas. sets and the class of μ-meas. sets (for any Borel prob. measure μ) are closed under operation $\mathcal{A}$. This is done using the notion of an envelope for a σ-algebra.

Def. For a σ-algebra $\mathcal{S}$ of subsets of X, call a set $A \subseteq X$ $\mathcal{S}$-small if $\mathrm{Powerset}(A) \subseteq \mathcal{S}$.

Examples. $\mathrm{MEAG} \overset{=}{\subseteq} \mathrm{BM}$-small and $\mathrm{NULL}_\mu \overset{=}{\subseteq} \mathrm{MEAS}_\mu$-small, one can show using Axiom of Choice that these inclusions are equalities.